A Generalization of the Hopf's Lemma for the 1-D Moving-Boundary Problem for the Fractional Diffusion Equation and its Application to a Fractional Free-Boundary Problem.

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Abstract: This paper deals with a theoretical mathematical analysis of a one-dimensional-moving-boundary problem for the time-fractional diffusion equation, where the time-fractional derivative of order $\alpha \in (0,1)$ is taken in the Caputo's sense. A generalization of the Hopf's lemma is proved, and then this result is used to prove a monotonicity property for the free-boundary when a fractional free-boundary Stefan problem is considered.

1 Introduction

The development of the fractional calculus dates from the XIX century. Mathematicians as Lacroix, Abel, Liouville, Riemann and Letnikov attemped to establish a definition of fractional derivative. But the definition given by Caputo in 1967, was an open door to the beginning of the physics applications, while the previous definitions enabled a great theoretical development.

The research on the theory of fractional differential equations has begun to develop recently, and in the past decades many authors pointed out that derivatives and integrals of non-integer order are very useful in describing the properties of various real-world materials such as polymers or some types of non-homogeneous solids. The trend indicates that the new fractional order models are more suitable than integer order models previously used, since fractional derivatives give us an excellent tool for describing properties of memory and heritage of various materials and processes. Works in this direction are [1, 5, 8, 13, 24].

This paper deals with the fractional diffusion equation (here in after FDE), obtained from the standard diffusion equation by replacing the first order time-derivative by a fractional derivative of order $\alpha > 0$ in the Caputo's sense:

$$_0D_t^\alpha u(x,t) = \lambda^2 \, u_{xx}(x,t), \quad -\infty < x < \infty, \ t > 0, \ 0 < \alpha < 1,$$

where the fractional derivative in the Caputo's sense of arbitrary order $\alpha > 0$ is given by

$$_aD^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n \\ f^{(n)}(t), & \alpha = n. \end{cases}$$

where $n \in \mathbb{N}$ and Γ is the Gamma function defined by $\Gamma(x) = \int_0^\infty w^{x-1}e^{-w}dw$.

Whereas that the one-dimensional heat equation has become the paradigm for the all-embracing study of parabolic partial differential equations, linear or nonlinear (see Cannon [4]), the FDE plays a similar role in fractional parabolic operators.

The FDE has been treated by a number of authors (see [9, 14, 17, 19, 21]) and, among the several applications that have been studied, Mainardi [20] studied the application to the theory of linear viscoelasticity.

Generalizations of the maximum principle for initial-boundary-value problems associated to the time-fractional diffusion equations were given by Luchko in [17] and [16], and uniqueness results there were obtained.

Eberhard Frederich Ferdinand Hopf was an Austrian mathematician who made significant contributions in differential equations, topology and ergodic theory. One of his most important works are related to the strong maximum principle for partial differential equations of elliptic type ([11]).

In his work [12], an important theorem related to the sign of the outside directional derivative of a function that is a solution to an elliptic partial differential inequality is proved. This theorem was proved later for partial differential operators of parabolic type by A. Friedman [7] and R. Viborni [25] separately. A weak adaptation of this theorem can be founded in [4], named Hopf's Lemma, and my propose is to generalize it for the FDE.

That is, under certain conditions that will be enunciated later, if u is a solution of the following problem

$$
\begin{cases}
(i) & {}_0D_t^\alpha u(x,t) = \lambda^2\, u_{xx}(x,t) & s_1(t) < x < s_2(t),\ 0 < t \le T,\ 0 < \alpha < 1 \\
(ii) & u(s_1(t),t) = g(t) & 0 < t \le T \\
(iii) & u(s_2(t),t) = h(t) & 0 < t \le T \\
(iv) & u(x,0) = f(x) & a \le x \le b
\end{cases}
\tag{1}
$$

where s_1 and s_2 are given, then, if u assumes its maximum in a boundary point, let us say $(s_2(t_0), t_0)$, it results that $u_x(s_2(t_0), t_0) > 0$.

2   A Fractional Hopf's Lemma

Let us consider the moving-boundary problem for the FDE defined in (1) where:

(H1) The curve s_1 is given and it is an upper Lipschitz continuous function.

(H2) The curve s_2 is given and it is a lower Lipschitz continuous function.

(H3) $s_1(0) = a$, $s_2(0) = b$, where $a \le b$, and condition (iv) of problem (1) is not considered if $a = b$.

(H4) $s_1(t) < s_2(t)\ \forall t \in (0, T]$.

(H5) f is a non-negative continuous function in $[a, b]$.

(H6) g and h are non-negative continuous functions in $(0, T]$.

We will consider the following two regions:
$D_T = \{(x,t)/s_1(t) < x < s_2(t),\ 0 < t \le T\}$ and the so-called parabolic boundary $\partial_p D_T = \{(s_1(t),t), 0 < t \le T\} \cup \{(s_2(t),t), 0 < t \le T\} \cup \{(x,0), a \le x \le b\}$.

Definition 1. *A function u is a solution of problem* (1) *if $u = u(x,t)$ verifies the conditions in* (1) *and*

1. u is defined in $[a_0, b_0] \times [0, T]$, where $a_0 = \min\{s_1(t),\ t \in [0,T]\}$ and $b_0 = \max\{s_2(t),\ t \in [0,T]\}$,

2. $u \in CW_{D_T} = C(D_T) \cap W^1((0,T)) \cap C_x^2(D_T)$, where $W^1((0,T)) = \{f \in C^1((0,T]) : f' \in L^1(0,T)\}$

3. u is continuous in $D_T \cup \partial_p D_T$ except perhaps at $(a,0)$ and $(b,0)$ where we will ask that

$$0 \le \liminf_{(x,t) \to (a,0)} u(x,t) \le \limsup_{(x,t) \to (a,0)} u(x,t) < +\infty$$

and

$$0 \le \liminf_{(x,t) \to (b,0)} u(x,t) \le \limsup_{(x,t) \to (b,0)} u(x,t) < +\infty.$$

Remark 1. *We ask u to be defined in $[a_0, b_0] \times [0,T]$ because the fractional derivative ${}_0D_t^\alpha u(x,t)$ involves the values of $u_t(x,\tau)$ for all τ in $[0,t]$. (See Figure 1).*

Remark 2. *This kind of problems has not been studied in depth yet, but taking into account the results obtained in [22] and [23], where some fractional Stefan problems has been solved explicitly, it is easy to check that the following problem*

$$\begin{cases} {}_0D_t^\alpha u(x,t) = u_{xx}(x,t) & 0 < x < t^{\alpha/2}, \ 0 < t \le T, \ 0 < \alpha < 1 \\ u(0,t) = B & 0 < t \le T \\ u(t^{\alpha/2}, t) = C & 0 < t \le T \end{cases} \tag{2}$$

admits the solution given by

$$u(x,t) = B + \frac{C - B}{1 - W\left(-1, -\frac{\alpha}{2}, 1\right)} \left[1 - W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right],$$

where $W\left(\cdot, -\frac{\alpha}{2}, 1\right)$ is the Wright function of parameters $\rho = -\frac{\alpha}{2}$ and $\beta = 1$ defined by

$$W(z; \rho; \beta) = \sum_{k=0}^\infty \frac{z^k}{k!\Gamma(\rho k + \beta)}, \quad z \in \mathbb{C}, \ \rho > -1, \beta \in \mathbb{R}. \tag{3}$$

The function $1 - W(-\cdot, -\frac{\alpha}{2}, 1)$ is the "fractional error function", so named because

$$\lim_{\alpha \nearrow 1} 1 - W(-x, -\frac{\alpha}{2}, 1) = erf\left(\frac{x}{2}\right)$$

(see [22] Theorem 4.1).

Hereinafter I will take $\lambda = 1$, I will call D^α to the fractional derivative in the Caputo's sense of extreme $a = 0$, ${}_0D_t^\alpha$, and L^α to the operator associated to the FDE

$$L^\alpha := \frac{\partial^2}{(\partial x)^2} - D^\alpha. \tag{4}$$

Proposition 1. *If u is a function such that $L^\alpha[u] > 0$ in D_T, then u can not attain its maximum at D_T.*

    Proof. Let us suppose that there exists $(x_0, t_0) \in D_T$ (that is $s_1(t_0) < x_0 < s_2(t_0)$, $0 < t_0 \le T$), such that u attains its maximum at (x_0, t_0).
Due to the extremum principle for the Caputo derivative (see [18]), we have that $D_t^\alpha u(x_0, t_0) \ge 0$.
On the other hand, since $u \in C_x^2(D_T)$, $\frac{\partial^2 u}{\partial x^2}(x_0, t_0) \le 0$. Then $L^\alpha[u](x_0, t_0) \le 0$, which is a contradiction.
$\square$

Corollary 1. *If u is a function such that $L^\alpha[u] < 0$ in D_T, then u can not attain its minimum in D_T.*

It is easy to adapt to the moving-boundary problem (1) results obtained in [16] for initial-boundary-value problems associated to the generalize FDE. For this reason we omit the proof of the following result.

Theorem 1. *Let $u \in CW_{D_T}$ be a solution of (1). Then either*

$$u(x,t) \geq 0 \, \forall \, (x,t) \in \overline{D_T} \quad \text{or } u \text{ attains its negative minimum on } \partial_p D_T.$$

Let us enunciate the main result of this paper.

Theorem 2. *Let $u \in CW_{D_T}$ be a solution of problem (1) satisfying the hypotheses $(H1) - (H6)$.*

1. *If there exists $t_0 > 0$ such that*

$$u(s_2(t_0), t_0) = M = \sup_{\partial_p D_T} u, \tag{5}$$

   and,

$$\text{exists } \delta > 0 \text{ such that } |s_1(t_0) - s_2(t_0)| \geq \delta \text{ and } u(x, t_0) < M \, \forall x \in (s_2(t_0) - \delta, s_2(t_0)), \tag{6}$$

   then

$$\liminf_{x \nearrow s_2(t_0)} \frac{u(x, t_0) - u(s_2(t_0), t_0)}{x - s_2(t_0)} > 0. \tag{7}$$

   If u_x exists at $(s_2(t_0), t_0)$, then

$$u_x(s_2(t_0), t_0) > 0. \tag{8}$$

2. *If there exists $t_0 > 0$ such that*

$$u(s_2(t_0), t_0) = m = \inf_{\partial_p D_T} u, \tag{9}$$

   and,

$$\text{exists } \delta > 0 \text{ such that } |s_1(t_0) - s_2(t_0)| \geq \delta \text{ and } u(x, t_0) > m \, \forall x \in (s_2(t_0) - \delta, s_2(t_0)), \tag{10}$$

   then

$$\limsup_{x \nearrow s_2(t_0)} \frac{u(x, t_0) - u(s_2(t_0), t_0)}{x - s_2(t_0)} < 0. \tag{11}$$

   If u_x exists at $(s_2(t_0), t_0)$, then

$$u_x(s_2(t_0), t_0) < 0. \tag{12}$$

Proof. I will prove 1. The proof of 2 is analogous.

Let us consider the function

$$w_\alpha(x, t) = \epsilon \left[1 - \exp\{-\mu(x - s_2(t_0))\} \frac{E_\alpha(\mu A t^\alpha)}{E_\alpha(\mu A t_0^\alpha)} \right] + M \tag{13}$$

where A, μ and ϵ will be determined and E_α is the Mittag-Leffler function defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}, \alpha > 0. \tag{14}$$

Note that $w_\alpha = M$ over the curve

$$\exp\{-\mu(x - s_2(t_0))\}\frac{E_\alpha(\mu At^\alpha)}{E_\alpha(\mu At_0^\alpha)} = 1. \tag{15}$$

Observe that the curve (15) is the graphic of the function

$$f(t) = \frac{1}{\mu}\ln\left(\frac{E_\alpha(\mu At^\alpha)}{E_\alpha(\mu At_0^\alpha)}\right) + s_2(t_0), \quad t \in (0, t_0] \tag{16}$$

Clearly $f(t_0) = s_2(t_0)$ and it is easy to check that

$$f \text{ is an increasing function if } \mu > 0. \tag{17}$$

Our next goal is to prove that there exists $t_1 < t_0$ such that $f(t) < s_2(t) \; \forall t \in (t_1, t_0)$.

Due to $(H2)$, s_2 is a lower Lipschitz continuous function, then there exists a constant $L > 0$ such that

$$\frac{s_2(t) - s_2(t_0)}{t - t_0} \le L \quad \forall \, 0 \le t \le t_0.$$

Therefore

$$s_2(t) \ge L(t - t_0) + s_2(t_0) \quad \forall \, 0 \le t \le t_0. \tag{18}$$

Taking into account that $E_\alpha(\mu At^\alpha) = \sum_{k=0}^\infty \frac{(\mu At^\alpha)^k}{\Gamma(\alpha k + 1)}$ is an uniform convergent series over compact sets contained in $(0, t_0]$ and that $z\Gamma(z) = \Gamma(z + 1) \; \forall \, z \in \mathbb{C}$, we have that

$$[E_\alpha(\mu At^\alpha)]' = \sum_{k=1}^\infty \frac{(\mu A)^k \alpha k t^{\alpha k - 1}}{\Gamma(\alpha k + 1)} = \sum_{k=0}^\infty \frac{(\mu A)^{k+1} t^{\alpha k + \alpha - 1}}{\Gamma(\alpha k + \alpha)} = \mu A t^{\alpha - 1} E_{\alpha,\alpha}(\mu At^\alpha),$$

where the function $E_{\alpha,\alpha}$ is the generalized Mittag-Leffler function of parameters $\rho = \beta = \alpha$ defined by

$$E_{\rho,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\rho k + \beta)}, \quad z \in \mathbb{C}, \rho > 0, \beta \in \mathbb{C}. \tag{19}$$

Then,

$$f'(t) = \frac{1}{\mu}\frac{1}{E_\alpha(\mu At^\alpha)}\frac{\mu A}{t^{1-\alpha}}E_{\alpha,\alpha}(\mu At^\alpha) = \frac{A}{t^{1-\alpha}}\frac{E_{\alpha,\alpha}(\mu At^\alpha)}{E_\alpha(\mu At^\alpha)}. \tag{20}$$

Now, let us define the function $H : \mathbb{R}_0^+ \to \mathbb{R} / H(t) = \frac{E_{\alpha,\alpha}(\mu At^\alpha)}{E_\alpha(\mu At^\alpha)}$.

H is a positive function and it is a quotient of continuous functions, where its denominator is grater than 1 in $[0, \infty)$, then H is continuous in $[0, \infty)$.
$H(0) = \frac{1}{\Gamma(\alpha)} > 0$ because $0 < \alpha < 1$.
$H(+\infty) = C > 0$ because it is a quotient of continuous functions with equal order in ∞ (see [10]).
Then, we can assure that there exists $m_0 > 0$ such that

$$H(t) \ge m_0 \quad \forall \, t \ge 0, \forall \, A, \mu > 0. \tag{21}$$

From (20) and (21), we have that

$$f'(t_0) \ge \frac{A}{t_0^{1-\alpha}}m_0. \tag{22}$$

Selcting $A > 0$ such that $\frac{A}{t_0^{1-\alpha}}m_0 > L$ we can assure that

$$f'(t_0) > L. \tag{23}$$

Lately, let be $\rho > 0$ / $f'(t_0) - \rho > L$. Due to the differentiability of f at t_0 we can assure that there exists $t_1 < t_0$ such that $\forall t \in (t_1, t_0)$,

$$L < f'(t_0) - \rho < \frac{f(t) - f(t_0)}{t - t_0} \Rightarrow f(t) < L(t - t_0) + f(t_0) = L(t - t_0) + s_2(t_0) \leq s_2(t)$$

Note that due to (6) and (17), we can select t_1 so that $s_1(t) < f(t) < s_2(t)$ $\forall t \in (t_1, t_0)$.

Now, let be $A(x_1, t_0)$ (where $x_1 = f(t_1)$), $B(s_2(t_0), t_0)$ and $C(x_1, t_1)$. Hypothesis (6) allows us to set t_1 again such that $x_1 \in (s_2(t_0) - \delta, s_2(t_0))$ and $u < M$ in $\overline{AC}$.
Let be $\mathcal{R}$ the region limited by $\overline{AB}$, $\overline{AC}$ and the portion of graph of f from B to C, which we will call $\widehat{CB}$.(See Figure 2)

The region $\mathcal{R}_{t_0} = \mathcal{R}^\circ \cup (\overline{AB} - \{A, B\})$ and its parabolic boundary $\partial_p \mathcal{R} = \overline{AC} \cup \widehat{CB}$ will be considered.

Next, we will analyze the behavior of u and w_α in the parabolic boundary $\partial_p \mathcal{R}$.
Let be $M_0 = \max_{t_1 \leq t \leq t_0} u(x_1, t)$. Because of the continuity of u, the hypothesis (6) and resetting t_1 if it is necessary, we can affirm that $M_0 < M$. Calling $\eta = M - M_0$, it yields that

$$u \leq M - \eta \quad \text{in } \overline{AC}. \tag{24}$$

$$u \leq M \quad \text{in } \widehat{CB}. \tag{25}$$

By the other side,
$$w_\alpha = M \quad \text{in } \widehat{CB}. \tag{26}$$

In $\overline{AC}$, considering that $E_\alpha(\mu A t^\alpha)$ is an increasing function, we have that,

$$w_\alpha(x_1, t) = \epsilon \left[1 - \exp\{-\mu(x_1 - s_2(t_0))\} \frac{E_\alpha(\mu A t^\alpha)}{E_\alpha(\mu A t_0^\alpha)}\right] + M \geq \epsilon \left[1 - \exp\{-\mu(x_1 - s_2(t_0))\}\right] + M. \tag{27}$$

Taking $\epsilon = \frac{\eta}{exp\{-\mu(x_1 - s_2(t_0))\} - 1}$, it results that

$$w_\alpha(x_1, t) \geq -\eta + M. \tag{28}$$

From [14] we have that
$$D^\alpha(E_\alpha(\mu A t^\alpha)) = \mu A E_\alpha(\mu A t^\alpha). \tag{29}$$

Now, applying the operator L^α to the function w_α and using (29) it yields that

$$L^\alpha[w_\alpha](x, t) = \epsilon \exp\{-\mu(x - s_2(t_0))\} \frac{E_\alpha(\mu A t^\alpha)}{E_\alpha(\mu A t_0^\alpha)}(\mu A - \mu^2) < 0 \quad \text{if } \mu = A + 1. \tag{30}$$

Finally, we define the function $z = w_\alpha - u$ in $\mathcal{R}$. Let us analyze the behavior of z in the parabolic boundary $\partial_p \mathcal{R}$.

From (24) and (28), $z \geq 0$ in $\overline{AC}$.

From (25) and (26), $z \geq 0$ in $\widehat{CB}$.

Also, $L^\alpha[z] = L^\alpha[w_\alpha] - L^\alpha[u] < 0$ in $\mathcal{R}_{t_0}$.

Applying Corollary 1, we can state that z cannot assume its minimum at $\mathcal{R}_{t_0}$. Then

$$z \geq 0 \quad \text{in } \mathcal{R}$$

In particular,

$$z(x, t_0) = w_\alpha(x, t_0) - u(x, t_0) \geq 0, \quad \forall \, x_1 \leq x \leq s_2(t_0). \tag{31}$$

Recalling that $u(s_2(t_0), t_0) = w_\alpha(s_2(t_0, t_0)) = M$, the next expression es equivalent to (31):

$$\frac{u(x, t_0) - u(s_2(t_0), t_0)}{x - s_2(t_0)} \geq \frac{w_\alpha(x, t_0) - w_\alpha(s_2(t_0, t_0))}{x - s_2(t_0)}. \tag{32}$$

Then

$$\liminf_{x \nearrow s_2(t_0)} \frac{u(x, t_0) - u(s_2(t_0), t_0)}{x - s_2(t_0)} \geq \liminf_{x \nearrow s_2(t_0)} \frac{w_\alpha(x, t_0) - w_\alpha(s_2(t_0, t_0))}{x - s_2(t_0)}.$$

But w_α is a differentiable function at $(s(t_0), t_0)$, then

$$\liminf_{x \nearrow s_2(t_0)} \frac{w_\alpha(x, t_0) - w_\alpha(s_2(t_0, t_0))}{x - s_2(t_0)} = (w_\alpha)_x(s_2(t_0), t_0) = \epsilon\mu = \epsilon(A + 1) > 0$$

and (7) holds.

Finally, if u_x exists at $(s_2(t_0), t_0)$, (32) implies that

$$u_x(s_2(t_0), t_0) \geq (w_\alpha)_x(s_2(t_0), t_0) > 0 \tag{33}$$

and (8) holds.

$\square$

The same result is valid if we consider s_1 instead of s_2.

Theorem 3. *Let $u \in CW_{D_T}$ be a solution of problem* (1) *satisfying the hypotheses* $(H1) - (H6)$.

1. *If there exists $t_0 > 0$ such that*

$$u(s_1(t_0), t_0) = M = \sup_{\partial_p D_T} u, \tag{34}$$

   and,

   exists $\delta > 0$ such that $|s_1(t_0) - s_2(t_0)| \geq \delta$ and $u(x, t_0) < M \, \forall x \in (s_1(t_0), s_1(t_0) + \delta)$, \tag{35}

   then

$$\limsup_{x \nearrow s_1(t_0)} \frac{u(x, t_0) - u(s_1(t_0), t_0)}{x - s_1(t_0)} < 0. \tag{36}$$

   If u_x exists at $(s_1(t_0), t_0)$, then

$$u_x(s_1(t_0), t_0) < 0. \tag{37}$$

2. *If there exists $t_0 > 0$ such that*

$$u(s_1(t_0), t_0) = m = \inf_{\partial_p D_T} u, \tag{38}$$

   and,

   exists $\delta > 0$ such that $|s_1(t_0) - s_2(t_0)| \geq \delta$ and $u(x, t_0) > m \, \forall x \in (s_1(t_0), s_1(t_0) + \delta)$, \tag{39}

   then

$$\liminf_{x \nearrow s_1(t_0)} \frac{u(x, t_0) - u(s_1(t_0), t_0)}{x - s_1(t_0)} > 0. \tag{40}$$

   If u_x exists at $(s_1(t_0), t_0)$, then

$$u_x(s_1(t_0), t_0) > 0. \tag{41}$$

3   An Application to Fractional Free-Boundary Stefan Problems.

Let us consider now the following fractional free-boundary Stefan problem for the FDE, where we have replaced the Stefan condition $\dfrac{ds(t)}{dt} = ku_x(s(t), t)$, $t > 0$, by the fractional Stefan condition

$$D^\alpha s(t) = -ku_x(s(t), t), \quad t > 0, \quad 0 < \alpha < 1.$$

$$\begin{cases}
(i) & D^\alpha u(x, t) = u_{xx}(x, t) & 0 < x < s(t),\, 0 < t < T,\, 0 < \alpha < 1,\, \lambda > 0 \\
(ii) & u(x, 0) = f(x) & 0 \le x \le b = s(0) \\
(iii) & u(0, t) = g(t) & 0 < t \le T \\
(iv) & u(s(t), t) = 0 & 0 < t \le T \\
(v) & D^\alpha s(t) = -ku_x(s(t), t) & 0 < t \le T, \quad k > 0 \text{ constant}
\end{cases} \tag{42}$$

Definition 2. *A pair $\{u, s\}$ is a solution of problem (42) if*

1. *u and s satisfies (42),*

2. *s is a continuous function in $[0, T]$ such that $s \in W(0, T) = \{f \in C^1((0, T]) \, / \, f' \in L^1(0, T)\}$.*

3. *u is defined in $[0, b_0] \times [0, T]$ where $b_0 = \max\{s(t), 0 \le t \le T\}$,*

4. *$u \in CW_{D_T}$.*

5. *u is continuous in $D_T \cup \partial_p D_T$ except perhaps at $(0, 0)$ and $(b, 0)$ where we will ask that*

$$0 \le \liminf_{(x,t)\to(0,0)} u(x, t) \le \limsup_{(x,t)\to(0,0)} u(x, t) < +\infty$$

and

$$0 \le \liminf_{(x,t)\to(b,0)} u(x, t) \le \limsup_{(x,t)\to(b,0)} u(x, t) < +\infty.$$

6. *There exists $u_x(s(t), t)$ for all $t \in (0, T]$.*

This kind of problems have been recently treated in [3, 6, 15, 22]), and our goal now is to prove the next theorem involving the monotonicity of the free boundary.

Theorem 4. *Let $\{u_1, s_1\}$ and $\{u_2, s_2\}$ be solutions of the fractional free-boundary Stefan problems (42) corresponding, respectively, to the data $\{b_1, f_1, g_1\}$ and $\{b_2, f_2, g_2\}$. Suppose that $b_1 < b_2$, $0 \le f_1 \le f_2$ and $0 \le g_1 \le g_2$. Then $s_1(t) < s_2(t) \forall\, t \in [0, T)$.*

Proof. s_1 and s_2 are continuous functions and $s_1(0) = b_1 < b_2 = s_2(0)$. Suppose that the set $A = \{t \in [0, T] \, / \, (s_1 - s_2)(t) = 0\} \ne \emptyset$, and let be $t_0 = \min A$. Due to the continuity of s_1 and s_2, $s_1(t_0) = s_2(t_0)$, and t_0 is the first t for which $s_1(t_0) = s_2(t_0)$.
Let be $h : [0, t_0] \to \mathbb{R}$ the function $h(t) = (s_1 - s_2)(t)$. h has the following properties:
(h-1) $h \in C^1(0, t_0] \cap C[0, t_0]$ (due to definition 2).
(h-2) $h(0) = b_1 - b_2 < 0$.
(h-3) h is a non positive function and $h(t_0) = 0$.
From (h-1)-(h-3), h attains its maximum value at t_0.
Using the estimate (eq. (12) of Theorem 2.1 of [2]), it results that

$$D^\alpha h(t_0) \ge \frac{h(t_0) - h(0)}{t_0^\alpha \Gamma(1 - \alpha)}. \tag{43}$$

Then,

$$D^\alpha h(t_0) \ge \frac{b_2 - b_1}{t_0^\alpha \Gamma(1 - \alpha)} > 0. \tag{44}$$

Taking into account the linearity of the Caputo's fractional derivative and that s_1 and s_2 satisfies the Stefan condition (42)-(v), (44) implies that

$$u_{2x}(s_2(t_0), t_0) - u_{1x}(s_1(t_0), t_0) > 0. \tag{45}$$

By the other hand, the function $w(x,t) = u_2(x,t) - u_1(x,t)$ is a solution of the following moving-boundary problem

$$\begin{cases} D^\alpha w(x,t) = w_{xx}(x,t) & 0 < x < s_1(t),\ 0 < t \leq t_0,\ 0 < \alpha < 1, \\ w(0,t) = (g_2 - g_1)(t) \geq 0 & 0 < t \leq t_0 \\ w(s_1(t),t) = u_2(s_1(t),t) & 0 < t \leq t_0 \\ w(x,0) = (f_2 - f_1)(x) \geq 0 & 0 \leq x \leq b_1 = s_1(0) \end{cases} \tag{46}$$

Applying Theorem 1 to u_2 in the region $\overline{D_{t_0}^2}$ where $D_{t_0}^2 = \{(x,t) \,/\, 0 < t \leq t_0,\ 0 < x < s_2(t)\}$, it results that $u_2(s_1(t),t) \geq 0$.

Now, applying again Theorem 1 to w in problem (46), it results that $w \geq 0$ in $\overline{D_{t_0}^1}$, where $D_{t_0}^1 = \{(x,t) \,/\, 0 < t \leq t_0,\ 0 < x < s_1(t)\}$.

Then we can state that w attains a minimum at $(s_1(t_0), t_0)$.

Now, if there exists $\epsilon > 0$ such that $w(x,t_0) > 0$ for all $x \in (s_1(t_0) - \epsilon, s_1(t_0))$, applying Theorem 2-2 we can conclude that $w_x(s_1(t_0), 0) < 0$. And then

$$u_{2x}(s_2(t_0), t_0) - u_{1x}(s_1(t_0), t_0) < 0, \tag{47}$$

which contradicts (46).

If, by contrast, we have sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ and for every $n \in \mathbb{N}$ there exists $x_n \in (s_1(t_0) - \epsilon_n, s_1(t_0))$ such that $w(x_n, t_0) = 0$, then

$$\lim_{n \to \infty} \frac{w(x_n, t_0) - w(s_1(t_0), t_0)}{\epsilon_n} = 0$$

and due to the existence of $w_x(s_1(t_0), t_0)$ (definition 2-6), it results that $w_x(s_1(t_0), t_0) = 0$. Then

$$u_{2x}(s_2(t_0), t_0) - u_{1x}(s_1(t_0), t_0) = 0, \tag{48}$$

which contradicts (46) again.

This contradiction comes from assuming that there exists $t_0 > 0$ such that t_0 is the first t for which $s_1(t_0) = s_2(t_0)$. Therefore $s_1(t) < s_2(t) \forall\, t \in [0, T)$. $\qquad\square$

4 Conclusions

A one-dimensional moving-boundary problem for the time-fractional diffusion equation was presented, where the time-fractional derivative of order $\alpha \in (0,1)$ was taken in the Caputo's sense. Then, a generalization of the Hopf's lemma, involving the behavior of the partial x-derivative of the solution at a boundary point, was proved. This last result was used to prove a monotonicity property of the free-boundary, when a free-boundary Stefan problem was considered.

5 Acknowledgments

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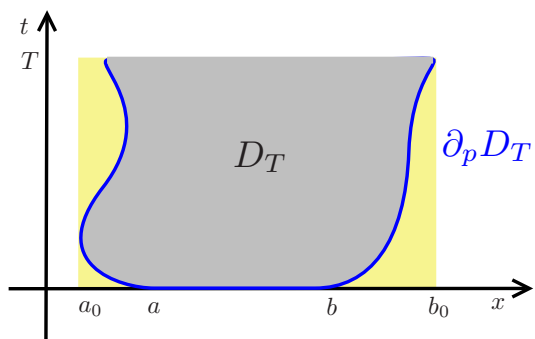[25] R. Viborni. On properties of solutions of some boundary value problems for equations of parabolic type. *Daklady Akad. Nauk SSSR (N.S.)*, 117:563–565, 1957.
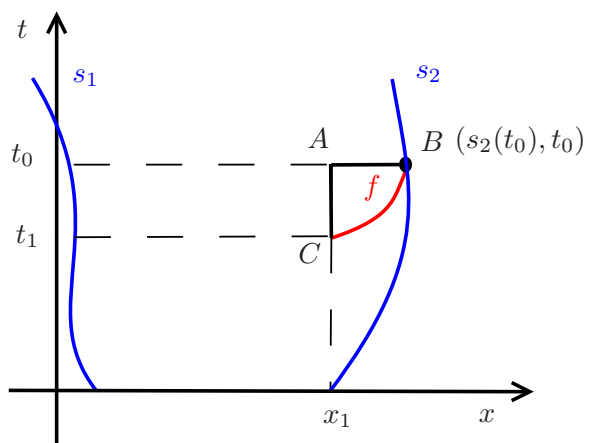
Figure 1: Figure 1

Figure 2: Figure 2